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A Fourier transform for sheaves on real tori Part I. The equivalence $\mathrm{Sky}(T) \simeq \mathrm{Loc}(\hat{T})$

U. Bruzzo [a,b,*], G. Marelli [a], F. Pioli [a,b]

[a] *Scuola Internazionale Superiore di Studi Avanzati (SISSA), Via Beirut 4, 34013 Trieste, Italy*
[b] *Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso 35, 16146 Genova, Italy*

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Abstract

As a first step toward a theory of a real Fourier transform for sheaves on Calabi–Yau manifolds fibred in special Lagrangian tori, we explicitly construct the functors which establish the equivalence between the category of skyscraper sheaves of finite-dimensional vector spaces on a real torus T, and the category of local systems (locally free sheaves of $\mathbb{C}$-modules of finite rank) on the dual torus $\hat{T}$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A “Fourier” transform, mapping coherent sheaves on an abelian variety X to coherent sheaves on the dual variety $\hat{X}$, was introduced by Mukai [13] (more precisely, the Fourier–Mukai transform is a functor $D(X) \to D(\hat{X})$, where $D(X)$ is the derived category of coherent sheaves of $\mathcal{O}_X$-modules). A relative Fourier–Mukai transform for elliptic surfaces was developed in [3,4,6,9] and was shown to play a role in the description of mirror symmetry for K3 surfaces [2,3]; this Fourier–Mukai transform describes the transformation of D-branes under mirror symmetry, as suggested by the Strominger–Yau–Zaslow conjecture [14].

In the case of Calabi–Yau threefolds, which, again in accordance with the Strominger–Yau–Zaslow conjecture, are supposed to be fibred in (special Lagrangian) real 3-tori (see

* Corresponding author.
E-mail addresses: bruzzo@sissa.it (U. Bruzzo), marelli@sissa.it (G. Marelli), pioli@fm.sissa.it (F. Pioli).

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[8] and references therein for a mathematical treatment), a similar description should be provided by a "real" relative Fourier transform. Here a great difficulty is offered by the presence of singular fibres. As a first approximation one can consider the simplified case of smooth fibrations (of course in this case the Calabi–Yau manifold is non-compact).

In this paper we take a first step toward the construction of this transform by defining the functors which establish the equivalence between the category $\mathrm{Sky}(T)$ of skyscraper sheaves of finite-dimensional vector spaces on a real torus T, and the category $\mathrm{Loc}(\hat{T})$ of local systems (locally free sheaves of $\mathbb{C}$-modules of finite rank) on the dual torus $\hat{T}$ (this results parallels the one holding in the holomorphic category, cf. [12]). This provides a proof for the claim forwarded by Arinkin and Polishchuk in [1]. In this connection, it should be remarked that this result (Theorem 3.12 in this paper) is just stated in Ref. [1], and the inverse functor is not constructed there.

Further papers will deal with the relative case (in this connection, one should notice that some results are already contained in [11]). In particular, in a second part of this series we shall describe a relative real Fourier–Mukai transform which establishes an equivalence between the category of local systems supported by Lagrangian submanifolds of a symplectic family of real tori X, and a category whose objects are holomorphic families of relatively flat vector bundles on the dual family $\hat{X}$. This should be related to the assumed equivalence between the Fukaya category of X and a suitable deformation of the derived category of coherent sheaves on $\hat{X}$.

This paper is structured as follows. In Section 2 we offer a description of (smooth) $U(1)$ bundles on real tori in terms of their factor of automorphy which fully parallels the one available (in the holomorphic case) on complex tori (cf. [10]). This description of line bundles will be extensively used in the remainder of the paper. In Section 3.2 we describe two complexes which are naturally associated with the Poincaré sheaf. In Section 3.3 we introduce the functor $\mathrm{Sky}(T) \to \mathrm{Loc}(\hat{T})$. In Section 3.4 we introduce the functor $\mathrm{Loc}(\hat{T}) \to \mathrm{Sky}(T)$ and compute its action on local systems. This will require the computation of the cohomology of a complex associated with the Poincaré bundle and will form the main technical part of the paper.

2. Line bundles on real tori

2.1. Factors of automorphy

Let Λ be a g-dimensional lattice in a g-dimensional real vector space V, and let $T = V/\Lambda$ be the corresponding torus. Let $\mathrm{Pic}(T)$ denote the group of isomorphism classes of $U(1)$ bundles on T. The group $\mathrm{Pic}(T)$ is isomorphic to a group $P(\Lambda)$ we may associate with the lattice Λ in the following way. As a set, $P(\Lambda)$ is the set of pairs (A, χ), where $A \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})$ is an alternating two-form on Λ, and χ is a *semicharacter* for A, namely, a map $\chi : \Lambda \to U(1)$ such that

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)\,\mathrm{e}^{\mathrm{i}\pi A(\lambda,\mu)}$$

for all $\lambda, \mu \in \Lambda$. The group structure is the one given by

$$(A_1, \chi_1) \cdot (A_2, \chi_2) = (A_1 + A_2, \chi_1 \chi_2).$$

The isomorphism $\mathrm{Pic}(T) \simeq P(\Lambda)$ is the Appell–Humbert theorem for real tori. Via the isomorphism $\mathrm{Alt}^2(\Lambda, \mathbb{Z}) \simeq H^2(T, \mathbb{Z})$, the form A is to be identified with the first Chern class of L. In this way we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, U(1)) \to \mathrm{Pic}(T) \xrightarrow{c_1} H^2(T, \mathbb{Z}) \to 0$$

and the kernel $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, U(1))$, whose elements are isomorphism classes of flat line bundles, is isomorphic to the dual torus $\hat{T} = V^\vee/\Lambda^\vee$ (here $V^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$). To every point $y \in \hat{T}$ there corresponds a flat line bundle L_y whose associated pair is

$$A_y = 0, \qquad \chi_y(\lambda) = \mathrm{e}^{2\mathrm{i}\pi y(\lambda)}.$$

The description of the bundle L by means of the pair (A, χ) allows one to give an explicit characterisation of the global sections of L. To this end one introduces the *factor of automorphy* of the pair (A, χ), defined as the map

$$a_L : V \times \Lambda \to U(1), \qquad a_L(x, \lambda) = \chi(\lambda)\, \mathrm{e}^{\mathrm{i}\pi A(x,\lambda)}$$

(here A has been extended to $V \times V$ in the natural way).

Proposition 2.1. *Let L be a line bundle on T, corresponding to the pair $(A, \chi) \in P(\Lambda)$. The global sections of L are in a one-to-one correspondence with the smooth functions $s : V \to \mathbb{C}$ satisfying the automorphy condition*

$$s(x + \lambda) = a_L(x, \lambda) s(x)$$

for all $x \in V$, $\lambda \in \Lambda$.

Proof. The proof is a (simplified) replica of the one holding in the case of complex tori [10] and will, therefore, be omitted. □

The action of an automorphism of L changes the factor of automorphy; an automorphism of L is induced by a map $\phi : V \to U(1)$, and the new factor of automorphy is

$$a_L(x, \lambda)' = \phi(x + \lambda) a_L(x, \lambda) \phi(x)^{-1}.$$

2.2. The Poincaré bundle

Now we use these tools to describe the *Poincaré bundle* $\mathcal{P}$ on the product $T \times \hat{T}$. The line bundle $\mathcal{P}$ is associated with the pair $(A, \chi) \in P(\Lambda \times \Lambda^\vee)$, where

$$A((\lambda_1, \mu_1), (\lambda_2, \mu_2) = \mu_1(\lambda_2) - \mu_2(\lambda_1), \qquad \chi(\lambda, \mu) = \mathrm{e}^{\mathrm{i}\pi \mu(\lambda)}.$$

The corresponding factor of automorphy is

$$a_{\mathcal{P}}(x, y, \lambda, \mu) = \mathrm{e}^{\mathrm{i}\pi [y(\lambda) - \mu(x) - \mu(\lambda)]}.$$

It is convenient to apply the automorphism induced by the map

$$\phi : V \times V^\vee \to U(1), \qquad \phi(x, y) = \mathrm{e}^{\mathrm{i}\pi y(x)}$$

thus obtaining a new factor of automorphy

$$a'_{\mathcal{P}}(x, y, \lambda, \mu) = \mathrm{e}^{2\mathrm{i}\pi y(\lambda)}. \tag{1}$$

This description of the Poincaré bundle shows explicitly that $\mathcal{P}_{|T\times\{y\}} \simeq L_y$.

Let $\nabla_{\mathcal{P}}$ be the Levi-Civita connection of $\mathcal{P}$. Its connection form $\mathbb{A}$ is written in the gauge where the factor of automorphy of $\mathcal{P}$ has the form (1) as

$$\mathbb{A} = -2\mathrm{i}\pi \sum_{j=1}^{g} x^j \, \mathrm{d}y_j, \tag{2}$$

where $(x^1, \ldots, x^g)$ are flat coordinates on T and $(y_1, \ldots, y_g)$ are dual flat coordinates on $\hat{T}$. The restriction $\nabla_{\mathcal{P}_{|T\times\{y\}}}$ is the Levi-Civita connection of L_y.

If we act on $a_{\mathcal{P}}$ with the automorphism $\phi(x, y) = \mathrm{e}^{-\mathrm{i}\pi y(x)}$ we obtain the factor of automorphy $a''_{\mathcal{P}}(x, y, \lambda, \mu) = \mathrm{e}^{-2\mathrm{i}\pi \mu(x)}$ which shows that, after the identification $\hat{T} \simeq T$, the dual bundle $\mathcal{P}^\vee$ is a Poincaré bundle for $\hat{T} \times T$.

3. The absolute case

3.1. Some relevant categories

For every real torus T we shall consider the category $\mathrm{Mod}(\mathbb{C}_T)$ of $\mathbb{C}_T$-modules, where $\mathbb{C}_T$ is the constant sheaf on T, and two full subcategories, namely,

1. the subcategory $\mathrm{Sky}(T)$ of skyscrapers of total finite length (i.e., $\dim H^0(T, M) < \infty$ for all $M \in \mathrm{Ob}(\mathrm{Sky}(T))$);
2. the subcategory $\mathrm{Loc}(T)$ of local systems, i.e., locally free $\mathbb{C}_T$-modules of finite rank.

   We shall also need to consider the following categories:
3. the category $\mathrm{Mod}(\mathcal{C}_T^\infty)$ of $\mathcal{C}_T^\infty$-modules, where $\mathcal{C}_T^\infty$ is the sheaf of germs of C^∞ $\mathbb{C}$-valued functions on T;
4. the category $\mathrm{Vect}_0(T)$ of flat vector bundles on T. Objects in this category may be regarded as pairs (E, ∇), where E is a smooth complex vector bundle and $\nabla : E \to E \otimes \Omega^1_T$ is a flat connection on it (Ω^1_T is the sheaf of differential 1-forms on T).[1]

[1] One should notice that one can consider connections on *any* $\mathcal{C}_T^\infty$-module $\mathcal{E}$, not just locally free ones; a connection is a map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_T$ satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f.$$

More intrinsically, a connection on $\mathcal{E}$ is a splitting of the exact sequence

$$0 \to \mathcal{E} \otimes \Omega^1_T \to \mathcal{J}(\mathcal{E}) \to \mathcal{E} \to 0,$$

where $\mathcal{J}(\mathcal{E})$ is the first jet extension of the sheaf $\mathcal{E}$.

A morphism $(E_1, \nabla_1) \to (E_2, \nabla_2)$ is a morphism $\phi : E_1 \to E_2$ of $\mathcal{C}^\infty_T$-modules which is compatible with the connections, i.e., $\nabla_2 \circ \phi = (\phi \otimes 1) \circ \nabla_1$.

There are two naturally defined functors. The first one maps $\mathrm{Mod}(\mathbb{C}_T)$ into $\mathrm{Mod}(\mathcal{C}^\infty_T)$, and its action on the objects is

$$M \mapsto M \otimes_{\mathbb{C}_T} \mathcal{C}^\infty_T.$$

The second functor maps $\mathrm{Loc}(T)$ into $\mathrm{Vect}_0(T)$, and with a local system $\mathfrak{S}$ associates the vector bundle $E = \mathfrak{S} \otimes_{\mathbb{C}_T} \mathcal{C}^\infty_T$ and the connection defined by

$$\nabla(s \otimes f) = s \otimes \mathrm{d}f,$$

where d is the exterior differential. In both cases the action of the functors on morphisms is naturally defined. The second functor establishes an equivalence of categories between $\mathrm{Loc}(T)$ and $\mathrm{Vect}_0(T)$ (cf. [7]); its inverse maps the pair (E, ∇) to the $\mathbb{C}_T$-module $\ker \nabla$.

3.2. *Complexes associated with the Poincaré sheaf*

We turn now our attention to the Poincaré sheaf $\mathcal{P}$ on $T \times \hat{T}$, where T is a real torus of any dimension g. We denote by p, $\hat{p}$ the projections onto the two factors of $T \times \hat{T}$. To simplify notation we shall set

$$\Omega^{m,n} = p^*\Omega^m_T \otimes_{\mathcal{C}^\infty_{T\times\hat{T}}} \hat{p}^*\Omega^n_{\hat{T}},$$

where p^* denotes the pullback of $\mathcal{C}^\infty$-modules, i.e.,

$$p^*\mathcal{E} = p^{-1}\mathcal{E} \otimes_{p^{-1}\mathcal{C}^\infty_T} \mathcal{C}^\infty_{T\times\hat{T}},$$

and similarly for $\hat{p}^*$.

The Levi-Civita connection $\nabla_{\mathcal{P}}$ of $\mathcal{P}$ has Künneth splitting into two operators

$$\nabla_1 : \mathcal{P} \to \mathcal{P} \otimes \Omega^{1,0}, \qquad \nabla_2 : \mathcal{P} \to \mathcal{P} \otimes \Omega^{0,1}$$

both squaring to zero (but their anticommutator is the curvature of $\nabla_{\mathcal{P}}$). The action of ∇_1, ∇_2 on functions is locally written in the form

$$\nabla_1 f = \sum_{j=1}^{g} \frac{\partial f}{\partial x^j} \mathrm{d}x^j, \qquad \nabla_2 f = \sum_{j=1}^{g} \left(\frac{\partial f}{\partial y_j} - 2\mathrm{i}\pi x^j f \right) \mathrm{d}y_j. \tag{3}$$

Let $\mathcal{E}$ be a $\mathcal{C}^\infty_T$-module with a flat connection ∇. By pulling the pair $(\mathcal{E}, \nabla)$ back to $T \times \hat{T}$ and coupling it with the pair $(\mathcal{P}, \nabla_1)$ we obtain a complex

$$0 \to \ker \nabla_1^E \to p^*\mathcal{E} \otimes \mathcal{P} \xrightarrow{\nabla_1^E} p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{1,0} \xrightarrow{\nabla_1^E} p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{2,0} \to \cdots.$$

Since locally the operator ∇_1^E coincides with the exterior differential, this sheaf complex is exact, and is a fine resolution of the sheaf $\ker \nabla_1^E$. Thus we obtain an isomorphism

$$H^i(T \times \hat{T}, \ker \nabla_1^E) \simeq H^i(\Gamma(p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{\bullet,0}), \nabla_1^E), \quad i \geq 0$$

between the cohomology of the sheaf ker ∇_1^E and the cohomology of the complex $\Gamma(p^*\mathcal{E}\otimes\mathcal{P}\otimes\Omega^{\bullet,0})$ (where Γ is the global sections functor) acted upon by the differential ∇_1^E.

The same results hold for the operator ∇_2. The cohomology of the complex $(\Gamma(\mathcal{P}^\vee\otimes\Omega^{0,\bullet}),\nabla_2)$ will be computed in Section 3.4.

3.3. The functor Sky$(T)\to$ Loc$(\hat{T})$

We now define the functor $\mathcal{F}:\mathrm{Sky}(T)\to\mathrm{Loc}(\hat{T})$. Let M be a skyscraper of finite length on T. By additivity it suffices to consider the case where M is supported at a single point x of T. With M we associate the $\mathcal{C}_T^\infty$-module $\mathcal{M}=M\otimes_{\mathbb{C}_T}\mathcal{C}_T^\infty$. At first we produce an object $(E,\hat{\nabla})$ in $\mathrm{Vect}_0(\hat{T})$. Indeed one checks that the direct image

$$\hat{p}_*(p^*\mathcal{M}\otimes\mathcal{P})$$

is locally free of finite rank, so that it is the sheaf of sections of a vector bundle E with $\mathrm{rk}\,E=\mathrm{length}(M)$. Moreover, the operator ∇_2 naturally extends to $p^*\mathcal{M}\otimes\mathcal{P}$, and since the latter sheaf is supported on $\{x\}\times\hat{T}$, it induces an operator $\nabla:E\to E\otimes\Omega^1_{\hat{T}}$ which is a flat connection.

The object $\mathcal{F}(M)$ in $\mathrm{Loc}(\hat{T})$ is obtained by taking $\mathcal{F}(M)=\ker\nabla$. Standard checks show that this procedure does define a functor.

Example 3.1. Let $k(x)$ denote the one-dimensional skyscraper at $x\in T$. One has $\mathcal{F}(k(0))\simeq\mathbb{C}_{\hat{T}}$. Indeed, in this case we have $p^*\mathcal{M}\otimes\mathcal{P}\simeq\mathcal{C}^\infty_{\{0\}\times\hat{T}}$ and in view of Eqs. (2) and (3), the operator ∇_2 reduces on this sheaf to the exterior differential along the $\hat{T}$ direction. As a consequence $(E,\nabla)=(\mathcal{C}^\infty_{\hat{T}},d)$, and $\mathcal{F}(k(0))=\ker d\simeq\mathbb{C}_{\hat{T}}$.

For every $x\in T$ let t_x be the associated translation, $t_x(x')=x+x'$. Moreover, identify $\hat{T}$ with T. The following result is easily proved.

Proposition 3.2. *Regard $\mathcal{F}$ as taking values in* $\mathrm{Vect}_0(\hat{T})$. *For every $x\in T$ and $M\in\mathrm{Ob}(\mathrm{Sky}(T))$ there is an isomorphism $\mathcal{F}(t_x^{-1}M)\simeq L_{-x}\otimes\mathcal{F}(M)$.*

As a consequence, in view of Example 3.1, we have

Corollary 3.3. *For every $x\in T$ one has $\mathcal{F}(k(x))\simeq L_{-x}$.*

This defines the action of the functor $\mathcal{F}$ on the whole category Sky(T).

3.4. The functor Loc$(\hat{T})\to$ Sky(T)

It is not clear how to define an inverse for the functor $\mathcal{F}$ by means of the adjunction theory for $\mathbb{C}$-modules. In this section we shall rather give a direct construction of a functor $\hat{\mathcal{F}}:\mathrm{Loc}(\hat{T})\to\mathrm{Sky}(T)$ which inverts the functor $\mathcal{F}$. We shall construct the functor by starting from objects in $\mathrm{Vect}_0(\hat{T})$. If (E,∇) is such an object, let $\mathcal{E}$ be the sheaf of sections of E. As we did in Section 3.3, but reverting the roles of T and $\hat{T}$, we consider on the sheaf $\hat{p}^*\mathcal{E} \otimes \mathcal{P}^\vee$ an operator ∇_2^E obtained by coupling (the pullback of) ∇ with the operator ∇_2. We shall eventually prove the following proposition.

Proposition 3.4.

1. $R^j p_* \ker \nabla_2^E = 0$ *for* $j = 0, \dots, g-1$;
2. *The sheaf* $R^g p_* \ker \nabla_2^E$ *is a skyscraper of finite length.*

The functor $\hat{\mathcal{F}}$ is defined as $\hat{\mathcal{F}}((E, \nabla)) = R^g p_* \ker \nabla_2^E$.

As a first step we compute the action of $\hat{\mathcal{F}}$ on the trivial line bundle, i.e., we take $\mathcal{E} = \mathcal{C}^\infty_{\hat{T}}$ and $\nabla = \mathrm{d}$. Thus we want to compute the sheaves $R^j p_* \ker \nabla_2$. To this end we shall study the presheaves

$$U \rightsquigarrow H^j(U \times \hat{T}, \ker \nabla_2) \simeq H^j((\mathcal{P}^\vee \otimes \Omega^{0,\bullet})(U \times \hat{T}), \nabla_2),$$

whose associated sheaves are exactly the sheaves we are interested in.

As a first result we have the following proposition.

Proposition 3.5. $H^0(U \times \hat{T}, \ker \nabla_2) = 0$ *for all open sets* $U \subset T$, *so that* $p_* \ker \nabla_2 = 0$.

Proof. An element of $H^0(U \times \hat{T}, \ker \nabla_2)$ restricted to $\{x\} \times \hat{T}$, with $x \in U$, yields a global section of L_x, which is zero unless $x = 0$. By a density argument we get the result. □

To compute the higher-order direct images we first consider the case $g = 1$.

Proposition 3.6. *If* $g = 1$, *then* $R^1 p_* \ker \nabla_2 \simeq k(0)$.

Proof. We compute the cohomology groups $H^1(U \times \hat{T}, \ker \nabla_2) \simeq H^1((\mathcal{P}^\vee \otimes \Omega^{0,\bullet})(U \times \hat{T}), \nabla_2)$. We represent T as $\mathbb{R} = /\mathbb{Z}\lambda$ with $\lambda \in \mathbb{R}$ and $\hat{T} = \mathbb{R}/\mathbb{Z}\mu$ with $\mu = 1/\lambda$. Let W be the inverse image of U in $\mathbb{R}$.

We work now in a gauge where the factor of automorphy of $\mathcal{P}^\vee$ is $\mathrm{e}^{2\mathrm{i}\pi\mu(x)}$, and the operator ∇_2 is the $\hat{T}$-part of the exterior differential. An element in $((\mathcal{P}^\vee \otimes \Omega^{0,1})(U \times \hat{T}), \ker \nabla_2)$ may be written as $\tau = t(x, y)\,\mathrm{d}y$, where t is a function on $W \times V^\vee$ satisfying the automorphy condition

$$t(x, y + \mu) = t(x, y)\,\mathrm{e}^{2\mathrm{i}\pi\mu(x)}.$$

If τ is a coboundary, $\tau = \nabla_2 s$, one has

$$s(x, y) = \int_0^y t(x, u)\,\mathrm{d}u + c(x).$$

The function s must satisfy the automorphy condition, which amounts to the following condition on c:

$$c(x)(1 - \mathrm{e}^{2\mathrm{i}\pi\mu(x)}) = -\int_0^\mu t(x, u)\,\mathrm{d}u. \tag{4}$$

If $0 \notin U$ this condition may be solved for c, so that $H^1(U \times \hat{T}, \ker \nabla_2) = 0$. Thus $R^1 p_* \ker \nabla_2$ is a skyscraper supported at $0 \in T$.

If $0 \in U$, the condition (4) may be solved if and only if

$$\int_0^\mu t(0, u)\, \mathrm{d}u = 0,$$

so that $H^1(U \times \hat{T}, \ker \nabla_2) \simeq \mathbb{C}$. This proves the claim. □

We move to the higher-dimensional case by means of a Künneth-type argument.

Proposition 3.7. *If* $\dim T = g$ *we have*

1. $R^j p_* \ker \nabla_2 = 0$ *for* $j = 0, \ldots, g-1$;
2. $R^g p_* \ker \nabla_2 \simeq k(0)$.

Proof. A choice of flat coordinates $(x^1, \ldots, x^g)$ on T fixes an isomorphism $T \simeq S^1 \times \cdots S^1$. The Poincaré sheaf $\mathcal{P}$ on $T \times \hat{T}$ is the tensor product of the pullbacks of the Poincaré sheaves $\mathcal{P}_i$ on the i factors of $T \times \hat{T}$, as one can check for instance by describing the Poincaré bundles by their factors of automorphy. Let $U \subset T$ be of the form $U = U_1 \times \cdots \times U_g$, where each U_i lies in a factor of $\hat{T}$. If $g = 2$, a word-by-word translation of the Künneth theorem for de Rham cohomology (cf., e.g. [5]) gives a decomposition

$$H^j(U \times \hat{T}, \ker \nabla_2) = \bigoplus_{m+n=j} H^m(U_1 \times S^1, \ker \nabla_2^1) \otimes H^n(U_2 \times S^1, \ker \nabla_2^2),$$

whence we have, by Proposition 3.5,

$$H^j(U \times \hat{T}, \ker \nabla_2) = 0 \quad \text{for} \;\; j = 0, 1, \qquad H^2(U \times \hat{T}, \ker \nabla_2) \simeq \mathbb{C}.$$

Induction on g then yields, for every g,

$$H^j(U \times \hat{T}, \ker \nabla_2) = 0 \quad \text{for} \;\; j = 0, \ldots, g-1, \qquad H^g(U \times \hat{T}, \ker \nabla_2) \simeq \mathbb{C}.$$

This proves both claims. □

So we have also obtained

$$H^j(T \times \hat{T}, \ker \nabla_2) = \begin{cases} 0 & \text{for } j = 0, \ldots, g-1, \\ \mathbb{C} & \text{for } j = g. \end{cases}$$

The $\mathcal{C}^\infty(T)$-module structure of the gth cohomology group is given by $f \cdot \alpha = f(0)\alpha$.

Remark 3.8. The difference in the definitions of the functors $\mathcal{F}$ and $\hat{\mathcal{F}}$ is only apparent; if M is a skyscraper on T, the operator ∇_1 vanishes on the sheaf $p^{-1}M \otimes \mathcal{P}$, so that $\mathcal{F}$ is formally identical with $\hat{\mathcal{F}}$.

3.5. The equivalence

Let $\mathfrak{L}_x$ be the local system corresponding to the line bundle L_x with its flat connection. In analogy with Proposition 3.2, we have

Proposition 3.9. $\hat{\mathcal{F}}(\mathfrak{L}_{-x} \otimes_{\mathbb{C}_{\hat{T}}} \mathfrak{S}) \simeq t_x^{-1}\hat{\mathcal{F}}(\mathfrak{S})$ *for every* $x \in T$ *and every local system* $\mathfrak{S}$ *on* $\hat{T}$.

Corollary 3.10. $\hat{\mathcal{F}}(\mathfrak{L}_{-x}) \simeq k(x)$ *for every* $x \in T$.

Remark 3.11. Since any flat vector bundle on a torus is a direct sum of flat line bundles (i.e., every local system on $\hat{T}$ is a direct sum of local systems of the type $\mathfrak{L}_x$), Corollary 3.10 completely describes the action of the functor $\hat{\mathcal{F}}$.

Corollaries 3.3 and 3.10 and Remark 3.11 eventually prove the following theorem.

Theorem 3.12. *The functors* $\mathcal{F}$, $\hat{\mathcal{F}}$ *are inverse to each other, and establish an equivalence between the categories* $\mathrm{Sky}(T)$ *and* $\mathrm{Loc}(\hat{T})$.

Again, any question related to the behaviour of morphisms under the functor $\hat{\mathcal{F}}$ is simply a matter of routine checks.

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